CSCI 7000 Fall 2023: Inclusion–Exclusion

Joshua A. Grochow

Released: October 10, 2023
Updated: October 17, 2023
Due: Monday ~~Oct 16~~ Oct 23, 2023

Read before Thursday 10/19

- Enumerative Combinatorics Vol. I Section 3.1 for background on partially ordered sets (6 pages). We won't really need all of this for Exercise 3 below, but I think it helps to get a feel for posets if you haven't seen them before or need some brushing up on them. Section 3.2 (building new posets from old) adds to that, but it's even more true that we won't need the material in Section 3.2.

Exercises

1. Generatingfunctionology Chapter 4 Exercise 9 (p. 159), reproduced verbatim here:

   > Let G be a graph of n vertices, and let positive integers x, λ be given. Let $P(\lambda; x; G)$ denote the number of ways of assigning one of λ given colors to each of the vertices of G in such a way that exactly x edges of G have both endpoints of the same color.
   >
   > Formulate the question of determining P as a sieve [inclusion-exclusion] problem with a suitable set of objects and properties. Find a formula for $P(\lambda; x; G)$, and observe that it is a polynomial in the two variables λ and x. The *chromatic polynomial* of G is $P(\lambda; 0; G)$.

   Addendum: The above is wording in a somewhat confusing way. $P(\lambda; x; G)$ is supposed to be an *ordinary generating function* in x and a polynomial in λ. That is, it is supposed to be

$$\sum_{e=0}^{\infty} x^e \cdot \#\{\lambda\text{-colorings in which exactly } e \text{ edges are monochromatic}\}.$$

Note that this is really a finite sum since once $e > |E(G)|$, the number of such colorings is 0.

2. Given non-negative integers k, n, d, find the number of non-negative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_k = n$$

such that all x_i satisfy $0 \leq x_i \leq d$.

3. (a) Suppose you have three sets A, B, C, with the property that $A \cap B = A \cap C = B \cap C$ (which therefore necessarily also equals $A \cap B \cap C$). So among all the possible intersections of the sets, rather than 7 nonempty sets (the maximum possible), we have only 4: A, B, C, and $A \cap B \cap C$. If we want to count $A \cup B \cup C$, write it as $|A \cup B \cup C| = a|A| + b|B| + c|C| + d|A \cap B \cap C|$. What are the coefficients a, b, c, d?

   (b) Given a finite partially ordered set P, define its *Möbius function* as $\mu(x, x) = 1$ for all $x \in P$, and, for all $x < y$ in P, $\sum_{z: x \leq z \leq y} \mu(x, z) = 0$. Given a function $f\colon P \to \mathbb{C}$ (any ring will do, but we use $\mathbb{C}$ for familiarity), define $g(x) := \sum_{x' \leq x} f(x')$ for all $x \in P$. Then show that

$$f(y) = \sum_{y' \leq y} g(y')\mu(y', y)$$

   for all $y \in P$. (This is called the *Möbius inversion formula*.)

   (c) Calculate the Möbius function for the poset from part (a). What do you notice about its relationship to the answer from part (a)?

   (d) Calculate the Möbius function for the poset of all subsets of $\{1, \ldots, n\}$. Calculate the Möbius inversion formula and observe that it is precisely the usual inclusion-exclusion formula.

4. (Stanley, *Enumerative Combinatorics*, Volume I, second edition, Chapter 2, Exercise 14). Let $A_k(n)$ denote the number of collections S of k subsets of $\{1, \ldots, n\}$ such that no element of S is a subset of another element of S. Show that $A_1(n) = 2^n$ and $A_2(n) = (1/2)(4^n - 2 \cdot 3^n + 2^n)$. Try to compute $A_k(n)$ for $k = 3, 4$. Can you see the pattern? See for how large a k can you get a general formula (as a function of n).

Resources

- van Lint & Wilson Chapter 10
- Generatingfunctionology Section 4.2 for a generating function view of inclusion–exclusion
- Generatingfunctionology p. 113 for average number of fixed points of a permutation via inclusion–exclusion
- Enumerative Combinatorics Vol. I Chapter 2 for inclusion–exclusion
- Enumerative Combinatorics Vol. I Section 3.7 for Möbius inversion
- Enumerative Combinatorics Vol. I Section 3.18 for a discussion of how the underlying poset of a counting problem tells us what kind of generating functions are "allowable."
- Bender & Goodman, *Amer. Math. Monthly*, 1975 give a nice exposition of Möbius inversion and some of its applications.